Notes on the function gsw_pt_second_derivatives(SA,CT)

This function, **gsw_pt_second_derivatives**(SA,CT), evaluates the second derivatives of $\hat{\theta}(S_A,\Theta)$, namely

$$\hat{\theta}_{S_A S_A}, \qquad \hat{\theta}_{\Theta\Theta} \qquad \text{and} \qquad \hat{\theta}_{S_A\Theta} . \tag{1}$$

These second derivatives are found by taking finite differences of the first derivatives $\hat{\theta}_{S_A}$ and $\hat{\theta}_{\Theta}$ obtained from **gsw_pt_first_derivatives**(SA,CT) over small increments of Absolute Salinity and potential temperature. The increments of Absolute Salinity are $\pm 0.001\text{ g kg}^{-1}$ and the increments of potential temperature are $\pm 0.01\,°\text{C}$. If the input Absolute Salinity is less than 0.001 g kg^{-1} care is taken to ensure that the Absolute Salinity is never less than zero in any call to **gsw_pt_first_derivatives**(SA,CT).

The three outputs of this function, **gsw_pt_second_derivatives**(SA,CT), remain well-behaved as the input Absolute Salinity approaches zero, including at $S_A = 0\text{ g kg}^{-1}$.

<u>References</u>

IOC, SCOR and IAPSO, 2010: *The international thermodynamic equation of seawater – 2010: Calculation and use of thermodynamic properties*. Intergovernmental Oceanographic Commission, Manuals and Guides No. 56, UNESCO (English), 196 pp. Available from http://www.TEOS-10.org

Here follows appendix A.12 of the TEOS-10 Manual (IOC *et al.*, 2010).

A.12 Differential relationships between η, θ, Θ and S_A

Evaluating the fundamental thermodynamic relation in the forms (A.11.6) and (A.11.12) and using the four boxed equations in appendix A.11, we find the relations

$$\begin{aligned}(T_0+t)\,d\eta + \mu(p)\,dS_A &= \frac{(T_0+t)}{(T_0+\theta)}\,c_p(0)\,d\theta + \big[\mu(p) - (T_0+t)\,\mu_T(0)\big]dS_A \\ &= \frac{(T_0+t)}{(T_0+\theta)}\,c_p^0\,d\Theta + \left[\mu(p) - \frac{(T_0+t)}{(T_0+\theta)}\,\mu(0)\right]dS_A .\end{aligned} \tag{A.12.1}$$

The quantity $\mu(p)\,dS_A$ is now subtracted from each of these three expressions and the whole equation is then multiplied by $(T_0+\theta)/(T_0+t)$ obtaining

$$(T_0+\theta)\,d\eta \;=\; c_p(0)\,d\theta - (T_0+\theta)\,\mu_T(0)\,dS_A \;=\; c_p^0\,d\Theta - \mu(0)\,dS_A . \tag{A.12.2}$$

From this follows all the following partial derivatives between η, θ, Θ and S_A,

$$\Theta_\theta\big|_{S_A} = c_p(S_A,\theta,0)/c_p^0, \qquad \Theta_{S_A}\big|_\theta = \big[\mu(S_A,\theta,0) - (T_0+\theta)\,\mu_T(S_A,\theta,0)\big]/c_p^0, \tag{A.12.3}$$

$$\Theta_\eta\big|_{S_A} = (T_0+\theta)/c_p^0, \qquad \Theta_{S_A}\big|_\eta = \mu(S_A,\theta,0)/c_p^0, \tag{A.12.4}$$

$$\theta_\eta\big|_{S_A} = (T_0+\theta)/c_p(S_A,\theta,0), \qquad \theta_{S_A}\big|_\eta = (T_0+\theta)\,\mu_T(S_A,\theta,0)/c_p(S_A,\theta,0), \tag{A.12.5}$$

$$\theta_\Theta\big|_{S_A} = c_p^0/c_p(S_A,\theta,0), \qquad \theta_{S_A}\big|_\Theta = -\big[\mu(S_A,\theta,0) - (T_0+\theta)\,\mu_T(S_A,\theta,0)\big]/c_p(S_A,\theta,0), \tag{A.12.6}$$

$$\eta_\theta\big|_{S_A} = c_p(S_A,\theta,0)/(T_0+\theta), \qquad \eta_{S_A}\big|_\theta = -\mu_T(S_A,\theta,0), \tag{A.12.7}$$

$$\eta_{\Theta}\big|_{S_{\rm A}} = c_p^0 \big/ \left(T_0 + \theta\right), \qquad\qquad \eta_{S_{\rm A}}\big|_{\Theta} = -\mu\left(S_{\rm A},\theta,0\right) \big/ \left(T_0 + \theta\right). \qquad \text{(A.12.8)}$$

The three second order derivatives of $\hat{\eta}\left(S_{\rm A},\Theta\right)$ are listed in Eqns. (P.14) and (P.15) of appendix P. The corresponding derivatives of $\hat{\theta}\left(S_{\rm A},\Theta\right)$, namely $\hat{\theta}_{\Theta}$, $\hat{\theta}_{S_{\rm A}}$, $\hat{\theta}_{\Theta\Theta}$, $\hat{\theta}_{S_{\rm A}\Theta}$ and $\hat{\theta}_{S_{\rm A}S_{\rm A}}$ can also be derived using Eqn. (P.13), obtaining

$$\hat{\theta}_{\Theta} = \frac{1}{\tilde{\Theta}_{\theta}}, \quad \hat{\theta}_{S_{\rm A}} = -\frac{\tilde{\Theta}_{S_{\rm A}}}{\tilde{\Theta}_{\theta}}, \quad \hat{\theta}_{\Theta\Theta} = -\frac{\tilde{\Theta}_{\theta\theta}}{\left(\tilde{\Theta}_{\theta}\right)^3}, \quad \hat{\theta}_{S_{\rm A}\Theta} = -\frac{\tilde{\Theta}_{\theta S_{\rm A}}}{\left(\tilde{\Theta}_{\theta}\right)^2} + \frac{\tilde{\Theta}_{S_{\rm A}}\tilde{\Theta}_{\theta\theta}}{\left(\tilde{\Theta}_{\theta}\right)^3}, \qquad \text{(A.12.9a,b,c,d)}$$

$$\text{and} \quad \hat{\theta}_{S_{\rm A}S_{\rm A}} = -\frac{\tilde{\Theta}_{S_{\rm A}S_{\rm A}}}{\tilde{\Theta}_{\theta}} + 2\,\frac{\tilde{\Theta}_{S_{\rm A}}}{\tilde{\Theta}_{\theta}}\,\frac{\tilde{\Theta}_{\theta S_{\rm A}}}{\tilde{\Theta}_{\theta}} - \left(\frac{\tilde{\Theta}_{S_{\rm A}}}{\tilde{\Theta}_{\theta}}\right)^2 \frac{\tilde{\Theta}_{\theta\theta}}{\tilde{\Theta}_{\theta}}, \qquad \text{(A.12.10)}$$

in terms of the partial derivatives $\tilde{\Theta}_{\theta}$, $\tilde{\Theta}_{S_{\rm A}}$, $\tilde{\Theta}_{\theta\theta}$, $\tilde{\Theta}_{\theta S_{\rm A}}$ and $\tilde{\Theta}_{S_{\rm A}S_{\rm A}}$ which can be obtained by differentiating the polynomial $\tilde{\Theta}\left(S_{\rm A},\theta\right)$ from the TEOS-10 Gibbs function.