Notes on the function gsw_pt_first_derivatives(SA,CT)

This function, **gsw_pt_first_derivatives**(SA,CT), evaluates the first derivatives of potential temperature with respect to Conservative Temperature and Absolute Salinity, as given in Eqn. (A.12.6) in the TEOS-10 Manual (IOC et al. (2010)), repeated here.

$$\theta_{\Theta}\big|_{S_\mathrm{A}} = \hat{\theta}_{\theta} = c_p^0 \big/ c_p\left(S_\mathrm{A},\theta,0\right), \tag{A.12.6a}$$

$$\theta_{S_\mathrm{A}}\big|_{\Theta} = \hat{\theta}_{S_\mathrm{A}} = -\left[\mu\left(S_\mathrm{A},\theta,0\right) - \left(T_0+\theta\right)\mu_T\left(S_\mathrm{A},\theta,0\right)\right] \big/ c_p\left(S_\mathrm{A},\theta,0\right), \tag{A.12.6b}$$

This function **gsw_pt_first_derivatives**(SA,CT) achieves this by calling another GSW function, **gsw_CT_first_derivatives**(SA,pt), because, from Eqn. (A.12.9a,b) (repeated below) the first derivatives $\hat{\theta}_{\theta}$ and $\hat{\theta}_{S_\mathrm{A}}$ are simply related to $\tilde{\Theta}_{\theta}$ and $\tilde{\Theta}_{S_\mathrm{A}}$.

$$\hat{\theta}_{\Theta} = \frac{1}{\tilde{\Theta}_{\theta}}, \quad \hat{\theta}_{S_\mathrm{A}} = -\frac{\tilde{\Theta}_{S_\mathrm{A}}}{\tilde{\Theta}_{\theta}}, \quad \hat{\theta}_{\Theta\Theta} = -\frac{\tilde{\Theta}_{\theta\theta}}{\left(\tilde{\Theta}_{\theta}\right)^3}, \quad \hat{\theta}_{S_\mathrm{A}\Theta} = -\frac{\tilde{\Theta}_{\theta S_\mathrm{A}}}{\left(\tilde{\Theta}_{\theta}\right)^2} + \frac{\tilde{\Theta}_{S_\mathrm{A}}\tilde{\Theta}_{\theta\theta}}{\left(\tilde{\Theta}_{\theta}\right)^3}, \tag{A.12.9a,b,c,d}$$

The code evaluates potential temperature and then calls **gsw_CT_first_derivatives**(SA,pt) to find $\tilde{\Theta}_{\theta}$ and $\tilde{\Theta}_{S_\mathrm{A}}$, and then these are used to find $\hat{\theta}_{\Theta}$ and $\hat{\theta}_{S_\mathrm{A}}$.

Hence, this function **gsw_pt_first_derivatives**(SA,CT) is essentially the following four lines of code.

```
pt = gsw_pt_from_CT(SA,CT);
[CT_SA, CT_pt] = gsw_CT_first_derivatives(SA,pt);
pt_CT = ones(size(CT_pt))./CT_pt;
pt_SA = - CT_SA.*pt_CT;
```

This function is well behaved at $S_\mathrm{A} = 0\ \mathrm{g\,kg^{-1}}$.

References

IOC, SCOR and IAPSO, 2010: *The international thermodynamic equation of seawater – 2010: Calculation and use of thermodynamic properties*. Intergovernmental Oceanographic Commission, Manuals and Guides No. 56, UNESCO (English), 196 pp. Available from http://www.TEOS-10.org

Here follows appendix A.12 of the TEOS-10 Manual (IOC *et al*., 2010).

A.12 Differential relationships between η, θ, Θ and S_{A}

Evaluating the fundamental thermodynamic relation in the forms (A.11.6) and (A.11.12) and using the four boxed equations in appendix A.11, we find the relations

$$\begin{aligned}(T_0+t)\,\mathrm{d}\eta+\mu(p)\,\mathrm{d}S_{\mathrm{A}} &= \frac{(T_0+t)}{(T_0+\theta)}\,c_p(0)\,\mathrm{d}\theta\;+\;\big[\mu(p)-(T_0+t)\,\mu_T(0)\big]\mathrm{d}S_{\mathrm{A}}\\ &= \frac{(T_0+t)}{(T_0+\theta)}\,c_p^0\,\mathrm{d}\Theta\;+\;\left[\mu(p)-\frac{(T_0+t)}{(T_0+\theta)}\mu(0)\right]\mathrm{d}S_{\mathrm{A}}\,.\end{aligned}\qquad\text{(A.12.1)}$$

The quantity $\mu(p)\,\mathrm{d}S_{\mathrm{A}}$ is now subtracted from each of these three expressions and the whole equation is then multiplied by $(T_0+\theta)/(T_0+t)$ obtaining

$$(T_0+\theta)\,\mathrm{d}\eta \;=\; c_p(0)\,\mathrm{d}\theta-(T_0+\theta)\,\mu_T(0)\,\mathrm{d}S_{\mathrm{A}} \;=\; c_p^0\,\mathrm{d}\Theta-\mu(0)\,\mathrm{d}S_{\mathrm{A}}\,. \qquad\text{(A.12.2)}$$

From this follows all the following partial derivatives between η, θ, Θ and S_{A},

$$\Theta_\theta\big|_{S_{\mathrm{A}}}=c_p(S_{\mathrm{A}},\theta,0)/c_p^0,\qquad \Theta_{S_{\mathrm{A}}}\big|_\theta=\big[\mu(S_{\mathrm{A}},\theta,0)-(T_0+\theta)\,\mu_T(S_{\mathrm{A}},\theta,0)\big]/c_p^0, \qquad\text{(A.12.3)}$$

$$\Theta_\eta\big|_{S_{\mathrm{A}}}=(T_0+\theta)/c_p^0,\qquad \Theta_{S_{\mathrm{A}}}\big|_\eta=\mu(S_{\mathrm{A}},\theta,0)/c_p^0, \qquad\text{(A.12.4)}$$

$$\theta_\eta\big|_{S_{\mathrm{A}}}=(T_0+\theta)/c_p(S_{\mathrm{A}},\theta,0),\qquad \theta_{S_{\mathrm{A}}}\big|_\eta=(T_0+\theta)\,\mu_T(S_{\mathrm{A}},\theta,0)/c_p(S_{\mathrm{A}},\theta,0), \qquad\text{(A.12.5)}$$

$$\theta_\Theta\big|_{S_{\mathrm{A}}}=c_p^0/c_p(S_{\mathrm{A}},\theta,0),\qquad \theta_{S_{\mathrm{A}}}\big|_\Theta=-\big[\mu(S_{\mathrm{A}},\theta,0)-(T_0+\theta)\,\mu_T(S_{\mathrm{A}},\theta,0)\big]/c_p(S_{\mathrm{A}},\theta,0), \qquad\text{(A.12.6)}$$

$$\eta_\theta\big|_{S_{\mathrm{A}}}=c_p(S_{\mathrm{A}},\theta,0)/(T_0+\theta),\qquad \eta_{S_{\mathrm{A}}}\big|_\theta=-\mu_T(S_{\mathrm{A}},\theta,0), \qquad\text{(A.12.7)}$$

$$\eta_\Theta\big|_{S_{\mathrm{A}}}=c_p^0/(T_0+\theta),\qquad \eta_{S_{\mathrm{A}}}\big|_\Theta=-\mu(S_{\mathrm{A}},\theta,0)/(T_0+\theta). \qquad\text{(A.12.8)}$$

The three second order derivatives of $\hat{\eta}(S_{\mathrm{A}},\Theta)$ are listed in Eqns. (P.14) and (P.15) of appendix P. The corresponding derivatives of $\hat{\theta}(S_{\mathrm{A}},\Theta)$, namely $\hat{\theta}_\Theta$, $\hat{\theta}_{S_{\mathrm{A}}}$, $\hat{\theta}_{\Theta\Theta}$, $\hat{\theta}_{S_{\mathrm{A}}\Theta}$ and $\hat{\theta}_{S_{\mathrm{A}}S_{\mathrm{A}}}$ can also be derived using Eqn. (P.13), obtaining

$$\hat{\theta}_\Theta=\frac{1}{\tilde{\Theta}_\theta},\quad \hat{\theta}_{S_{\mathrm{A}}}=-\frac{\tilde{\Theta}_{S_{\mathrm{A}}}}{\tilde{\Theta}_\theta},\quad \hat{\theta}_{\Theta\Theta}=-\frac{\tilde{\Theta}_{\theta\theta}}{\big(\tilde{\Theta}_\theta\big)^3},\quad \hat{\theta}_{S_{\mathrm{A}}\Theta}=-\frac{\tilde{\Theta}_{\theta S_{\mathrm{A}}}}{\big(\tilde{\Theta}_\theta\big)^2}+\frac{\tilde{\Theta}_{S_{\mathrm{A}}}\tilde{\Theta}_{\theta\theta}}{\big(\tilde{\Theta}_\theta\big)^3}, \qquad\text{(A.12.9a,b,c,d)}$$

$$\text{and}\quad \hat{\theta}_{S_{\mathrm{A}}S_{\mathrm{A}}}=-\frac{\tilde{\Theta}_{S_{\mathrm{A}}S_{\mathrm{A}}}}{\tilde{\Theta}_\theta}+2\,\frac{\tilde{\Theta}_{S_{\mathrm{A}}}}{\tilde{\Theta}_\theta}\frac{\tilde{\Theta}_{\theta S_{\mathrm{A}}}}{\tilde{\Theta}_\theta}-\left(\frac{\tilde{\Theta}_{S_{\mathrm{A}}}}{\tilde{\Theta}_\theta}\right)^2\frac{\tilde{\Theta}_{\theta\theta}}{\tilde{\Theta}_\theta}, \qquad\text{(A.12.10)}$$

in terms of the partial derivatives $\tilde{\Theta}_\theta$, $\tilde{\Theta}_{S_{\mathrm{A}}}$, $\tilde{\Theta}_{\theta\theta}$, $\tilde{\Theta}_{\theta S_{\mathrm{A}}}$ and $\tilde{\Theta}_{S_{\mathrm{A}}S_{\mathrm{A}}}$ which can be obtained by differentiating the polynomial $\tilde{\Theta}(S_{\mathrm{A}},\theta)$ from the TEOS-10 Gibbs function.