

Notes on the function gsw_geo_strf_dyn_height_pc(SA,CT,delta_p)

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In both z-coordinate and density-coordinate forward ocean models, the variables are often interpreted as being piecewise constant in the vertical, and this function, **gsw_geo_strf_dyn_height_pc**, calculates the dynamic height anomaly taking this piecewise-constant nature of Absolute Salinity and Conservative Temperature into account. The code uses the computationally-efficient 48-term rational function expression of McDougall *et al.* (2011) for the specific volume of seawater $\hat{v}(S_A, \Theta, p)$ in terms of Absolute Salinity S_A , Conservative Temperature Θ and pressure p . This 48-term rational function expression for density is discussed in McDougall *et al.* (2011) and in appendix A.30 and appendix K of the TEOS-10 Manual (IOC *et al.* (2010)). For dynamical oceanography we may take the 48-term rational function expression for density as essentially reflecting the full accuracy of TEOS-10.

In a so-called “z-coordinate” model the “thicknesses” of the layers are independent of latitude, longitude and time. Often the vertical coordinate is interpreted as pressure in these “z-coordinate” models, and in this case the “thickness” of the i th layer, $\text{delta_p} = p^{i+1} - p^i$ is spatially and temporally invariant. In density-coordinate models (often called “isopycnal models” or “layered models”) and also in hybrid-coordinate models, the “thickness” of the i th layer, $p^{i+1} - p^i$, varies with latitude, longitude and time. In these layered models the “thicknesses” of the least dense layers are often zero at many locations in the ocean, because these density surfaces have outcropped.

The pressure input to the function **gsw_geo_strf_dyn_height_pc** is the matrix $\text{delta_p} = p^{i+1} - p^i$ and the code can handle zero values of this layer “thickness” in several layers. The function calculates the pressures p_{mid} at the mid-pressure of each layer by summing the delta_p of the shallower layers and adding half of $\text{delta_p} = p^{i+1} - p^i$ at the i th layer. Taking the layer index i to increase from 1 in the upper-most layer, the Absolute Salinity and Conservative Temperature are constant at S_A^i and Θ^i in each layer, and the shallower and deeper pressures bounding the i th layer are p^i and p^{i+1} respectively.

The specific volume anomaly is defined with respect to $S_A = S_{SO} \equiv 35.165\ 04\ \text{g kg}^{-1}$ and $\Theta = 0^\circ\text{C}$ as

$$\hat{\delta}(S_A, \Theta, p) = \hat{v}(S_A, \Theta, p) - \hat{v}(S_{SO}, 0^\circ\text{C}, p), \quad (1)$$

and the thermodynamic identity

$$h_p|_{S_A, \Theta} = \hat{h}_p = \hat{v}, \quad (2)$$

is used to write the dynamic height anomaly Ψ at the midpoint pressure $\bar{p}^n = \frac{1}{2}[p^{n+1} + p^n]$ of layer n as

$$\begin{aligned} \Psi &= - \int_{R_0}^{\bar{p}^n} \hat{\delta}(S_A[p'], \Theta[p'], p') dP' \\ &= - \int_{R_0}^{\bar{p}^n} \hat{v}(S_A[p'], \Theta[p'], p') dP' + \int_{R_0}^{\bar{p}^n} \hat{v}(S_{SO}, \Theta = 0^\circ\text{C}, p') dP' \\ &= - \sum_{i=1}^{n-1} \left[\hat{h}(S_A^i, \Theta^i, p^{i+1}) - \hat{h}(S_A^i, \Theta^i, p^i) \right] \\ &\quad - \hat{h}(S_A^n, \Theta^n, \frac{1}{2}[p^{n+1} + p^n]) + \hat{h}(S_A^n, \Theta^n, p^n) \\ &\quad + \hat{h}(S_{SO}, \Theta = 0^\circ\text{C}, \frac{1}{2}[p^{n+1} + p^n]). \end{aligned} \quad (3)$$

Note the lower limit of the pressure integral of $\hat{v}(S_{SO}, 0^\circ\text{C}, p')$ is $\hat{h}(S_{SO}, \Theta = 0^\circ\text{C}, 0\text{dbar})$ which is zero (being c_p^0 times $\Theta = 0^\circ\text{C}$). The function **gsw_geo_strf_dyn_height_pc** evaluates the last of the three right-hand sides of Eqn. (3).

The enthalpy differences in Eqn. (3) for pairs of seawater parcels having the same Absolute Salinity and Conservative Temperature but different values of pressure are obtained from the function **gsw_enthalpy_diff** which has been designed to be computationally efficient. The last term in Eqn. (3) is found from the library function **gsw_enthalpy_SSO_0_p** which is a function of only pressure; this function is a streamlined version **gsw_enthalpy**, simplified because the Conservative Temperature argument is zero.

If the layer “thicknesses” $\text{delta_p} = p^{i+1} - p^i$ are spatially and temporally invariant then the output of the present function **gsw_geo_strf_dyn_height_pc** is the geostrophic streamfunction for the flow at pressure P (i.e. in layer n) with respect to the flow at the sea surface so that the two-dimensional gradient of Ψ in the P pressure surface is simply related to the difference between the horizontal geostrophic velocity \mathbf{v} at P and at the sea surface \mathbf{v}_0 according to

$$\mathbf{k} \times \nabla_p \Psi = f\mathbf{v} - f\mathbf{v}_0. \quad (4)$$

If however the layer “thicknesses” $\text{delta_p} = p^{i+1} - p^i$ are not spatially and temporally invariant then the output of the present function **gsw_geo_strf_dyn_height_pc** cannot be used by itself as a geostrophic streamfunction. Rather, it will be used as ingredient of the GSW function **gsw_geo_strf_isopycnal_pc** which does deliver an approximate geostrophic streamfunction for use in a variety of differently-defined isopycnal surfaces.

References

- IOC, SCOR and IAPSO, 2010: *The international thermodynamic equation of seawater – 2010: Calculation and use of thermodynamic properties*. Intergovernmental Oceanographic Commission, Manuals and Guides No. 56, UNESCO (English), 196 pp. Available from <http://www.TEOS-10.org> See section 3.27 and appendix A.30.
- McDougall T. J., P. M. Barker, R. Feistel and D. R. Jackett, 2011: A computationally efficient 48-term expression for the density of seawater in terms of Conservative Temperature, and related properties of seawater. submitted to *Ocean Science Discussions*.

Here follows section 3.27 of the TEOS-10 Manual (IOC *et al.* (2010)).

3.27 Dynamic height anomaly

The dynamic height anomaly Ψ with respect to the sea surface is given by

$$\Psi = - \int_{p_0}^P \hat{\delta}(S_A[p'], \Theta[p'], p') dP', \quad \text{where } \hat{\delta}(S_A, \Theta, p) = \hat{v}(S_A, \Theta, p) - \hat{v}(S_{SO}, 0^\circ\text{C}, p). \quad (3.27.1)$$

This is the geostrophic streamfunction for the flow at pressure P with respect to the flow at the sea surface and $\hat{\delta}$ is the specific volume anomaly. Thus the two-dimensional gradient of Ψ in the P pressure surface is simply related to the difference between the horizontal geostrophic velocity \mathbf{v} at P and at the sea surface \mathbf{v}_0 according to

$$\mathbf{k} \times \nabla_p \Psi = f\mathbf{v} - f\mathbf{v}_0. \quad (3.27.2)$$

The specific volume anomaly, $\hat{\delta}$ in the vertical integral in Eqn. (3.27.1) can be replaced with specific volume \hat{v} without affecting the isobaric gradient of the resulting streamfunction. That is, this substitution does not affect Eqn. (3.27.2) because the additional term is a function only of pressure. Traditionally it was important to use specific volume anomaly in preference to specific volume as it was more accurate with computer code which worked with single-precision variables. Since computers now regularly employ double-precision, this issue has been overcome and consequently either $\hat{\delta}$ or \hat{v} can be used in the integrand of Eqn. (3.27.1), so making it either the “dynamic height anomaly” or the “dynamic height”. As in the case of Eqn. (3.24.2), so also the dynamic height anomaly Eqn. (3.27.1) has not assumed that the gravitational acceleration is constant and so Eqn. (3.27.2) applies even when the gravitational acceleration is taken to vary in the vertical.

The dynamic height anomaly Ψ should be quoted in units of $\text{m}^2 \text{s}^{-2}$. These are the units in which the GSW Toolbox (appendix N) outputs dynamic height anomaly in the function `gsw_geo_strf_dyn_height(SA,CT,p,p_ref)`. When the last argument of this function, `p_ref`, is other than zero, the function returns the dynamic height anomaly with respect to a (deep) reference pressure `p_ref`, rather than with respect to P_0 (i.e. zero dbar sea pressure) as in Eqn. (3.27.1). In this case the lateral gradient of the streamfunction represents the geostrophic velocity difference relative to the (deep) p_{ref} pressure surface, that is,

$$\mathbf{k} \times \nabla_p \Psi = f\mathbf{v} - f\mathbf{v}_{\text{ref}}. \quad (3.27.3)$$

Note that the integration in Eqn. (3.27.1) of specific volume anomaly with pressure must be done with pressure in Pa (not dbar) in order to have the resultant isobaric gradient, $\nabla_p \Psi$, in the usual units, being the product of the Coriolis parameter (units of s^{-1}) and the velocity (units of m s^{-1}).