# Notes on the function gsw_beta_const_pt_t_exact(SA, t, p)

This function, **gsw_beta_const_pt_t_exact**(SA,t,p), evaluates the saline contraction coefficient with respect to Absolute Salinity $\beta^{\theta}$ at constant potential temperature $\theta$,

$$\beta^{\theta} = \frac{1}{\rho}\frac{\partial \rho}{\partial S_{\mathrm{A}}}\bigg|_{\theta,p} = -\frac{1}{v}\frac{\partial v}{\partial S_{\mathrm{A}}}\bigg|_{\theta,p} , \qquad (1)$$

with the input temperature being *in situ* temperature $t$. This function uses the full TEOS-10 Gibbs function $g(S_{\mathrm{A}},t,p)$ of IOC *et al.* (2010), being the sum of the IAPWS-09 and IAPWS-08 Gibbs functions. This function evaluates $\beta^{\theta}$ from the expression derived in appendix A.15 of the TEOS-10 Manual (IOC *et al.*, 2010), namely Eqn. (A.15.7), repeated here (and in **gsw_beta_const_pt_t_exact** we take the reference pressure to be $p_{\mathrm{r}} = 0$ dbar)

$$\beta^{\theta} = -\frac{g_{S_{\mathrm{A}}P}(S_{\mathrm{A}},t,p)}{g_P(S_{\mathrm{A}},t,p)} + \frac{g_{TP}(S_{\mathrm{A}},t,p)\left[g_{S_{\mathrm{A}}T}(S_{\mathrm{A}},t,p) - g_{S_{\mathrm{A}}T}(S_{\mathrm{A}},\theta,p_{\mathrm{r}})\right]}{g_P(S_{\mathrm{A}},t,p)\, g_{TT}(S_{\mathrm{A}},t,p)} . \qquad \text{(A.15.7)}$$

The first step in the code is to evaluate the potential temperature $\theta$ with respect to $p_{\mathrm{r}} = 0$ dbar. Both the terms $g_{S_{\mathrm{A}}T}(S_{\mathrm{A}},t,p)$ and $g_{S_{\mathrm{A}}T}(S_{\mathrm{A}},\theta,0)$ in the second part of this equation contain logarithmic singularities in the square root of Absolute Salinity, but these singularities exactly cancel in the square bracket in Eqn. (A.15.7). Hence, in the code this square bracket is not simply calculated by calling the appropriate derivatives of the Gibbs function, but rather, the polynomials representing both $g_{S_{\mathrm{A}}T}(S_{\mathrm{A}},t,p)$ and $g_{S_{\mathrm{A}}T}(S_{\mathrm{A}},\theta,p)$ are incorporated into the function **gsw_beta_const_pt_t_exact**(SA,t,p), and the logarithm terms are deliberately excluded. In this way, this function can be used with the input value of $S_{\mathrm{A}}$ being exactly zero.

References

IAPWS, 2008: Release on the IAPWS Formulation 2008 for the Thermodynamic Properties of Seawater. The International Association for the Properties of Water and Steam. Berlin, Germany, September 2008, available from www.iapws.org. This Release is referred to in the text as **IAPWS-08**.

IAPWS, 2009: Supplementary Release on a Computationally Efficient Thermodynamic Formulation for Liquid Water for Oceanographic Use. The International Association for the Properties of Water and Steam. Doorwerth, The Netherlands, September 2009, available from http://www.iapws.org. This Release is referred to in the text as **IAPWS-09**.

IOC, SCOR and IAPSO, 2010: *The international thermodynamic equation of seawater – 2010: Calculation and use of thermodynamic properties.* Intergovernmental Oceanographic Commission, Manuals and Guides No. 56, UNESCO (English), 196 pp. Available from http://www.TEOS-10.org

Here follows section 2.19 and appendix A.15 of the TEOS-10 Manual (IOC *et al.*, 2010).

## 2.19 Saline contraction coefficients

The saline contraction coefficient $\beta^{t}$ (sometimes also called the haline contraction coefficient) at constant *in situ* temperature $t$, is

$$\beta^t = \beta^t(S_A,t,p) = \frac{1}{\rho}\frac{\partial\rho}{\partial S_A}\bigg|_{T,p} = -\frac{1}{v}\frac{\partial v}{\partial S_A}\bigg|_{T,p} = -\frac{g_{S_A P}}{g_P}. \qquad (2.19.1)$$

The saline contraction coefficient $\beta^\theta$ at constant potential temperature $\theta$, is (see appendix A.15)

$$\beta^\theta = \beta^\theta(S_A,t,p,p_r) = \frac{1}{\rho}\frac{\partial\rho}{\partial S_A}\bigg|_{\theta,p} = -\frac{1}{v}\frac{\partial v}{\partial S_A}\bigg|_{\theta,p} = \frac{g_{TP}\left[g_{S_A T} - g_{S_A T}(S_A,\theta,p_r)\right] - g_{TT}\,g_{S_A P}}{g_P\,g_{TT}}, \qquad (2.19.2)$$

where $p_r$ is the reference pressure of $\theta$. One of the $g_{S_A T}$ derivatives in the numerator is evaluated at $(S_A,\theta,p_r)$ whereas all the other derivatives are evaluated at $(S_A,t,p)$.

The saline contraction coefficient $\beta^\Theta$ at constant Conservative Temperature $\Theta$, is (see appendix A.15)

$$\beta^\Theta = \beta^\Theta(S_A,t,p) = \frac{1}{\rho}\frac{\partial\rho}{\partial S_A}\bigg|_{\Theta,p} = -\frac{1}{v}\frac{\partial v}{\partial S_A}\bigg|_{\Theta,p} = \frac{g_{TP}\left[g_{S_A T} - (T_0+\theta)^{-1}\,g_{S_A}(S_A,\theta,0)\right] - g_{TT}\,g_{S_A P}}{g_P\,g_{TT}}. \qquad (2.19.3)$$

Note that Conservative Temperature $\Theta$ is defined only with respect to a reference pressure of 0 dbar as indicated in this equation. The $g_{S_A}$ derivative in the numerator is evaluated at $(S_A,\theta,0)$ whereas all the other derivatives are evaluated at $(S_A,t,p)$.

In the SIA computer software (appendix M) all three saline contraction coefficients are produced in units of $\text{kg kg}^{-1}$ while in the GSW Oceanographic Toolbox (appendix N) all three saline contraction coefficients are produced in units of $\text{kg g}^{-1}$ consistent with the preferred oceanographic unit for $S_A$ being $\text{g kg}^{-1}$.

## A.15 Derivation of the expressions for $\alpha^\theta$, $\beta^\theta$, $\alpha^\Theta$ and $\beta^\Theta$

This appendix derives the expressions in Eqns. (2.18.2) – (2.18.3) and (2.19.2) – (2.19.3) for the thermal expansion coefficients $\alpha^\theta$ and $\alpha^\Theta$ and the haline contraction coefficients $\beta^\theta$ and $\beta^\Theta$.

In order to derive Eqn. (2.18.2) for $\alpha^\theta$ we first need an expression for $\partial\theta/\partial T|_{S_A,p}$. This is found by differentiating with respect to *in situ* temperature the entropy equality $\eta(S_A,t,p) = \eta(S_A,\theta[S_A,t,p,p_r],p_r)$ which defines potential temperature, obtaining

$$\frac{\partial\theta}{\partial T}\bigg|_{S_A,p} = \frac{\eta_T(S_A,t,p)}{\eta_T(S_A,\theta,p_r)} = \frac{g_{TT}(S_A,t,p)}{g_{TT}(S_A,\theta,p_r)} = \frac{(T_0+\theta)}{(T_0+t)}\frac{c_p(S_A,t,p)}{c_p(S_A,\theta,p_r)}. \qquad (A.15.1)$$

This is then used to obtain the desired expression Eqn. (2.18.2) for $\alpha^\theta$ as follows

$$\alpha^\theta = \frac{1}{v}\frac{\partial v}{\partial\theta}\bigg|_{S_A,p} = \frac{1}{v}\frac{\partial v}{\partial T}\bigg|_{S_A,p}\left(\frac{\partial\theta}{\partial T}\bigg|_{S_A,p}\right)^{-1} = \frac{g_{TP}(S_A,t,p)}{g_P(S_A,t,p)}\frac{g_{TT}(S_A,\theta,p_r)}{g_{TT}(S_A,t,p)}. \qquad (A.15.2)$$

In order to derive Eqn. (2.18.3) for $\alpha^\Theta$ we first need an expression for $\partial\Theta/\partial t|_{S_A,p}$. This is found by differentiating with respect to *in situ* temperature the entropy equality $\eta(S_A,t,p) = \hat{\eta}(S_A,\Theta[S_A,t,p])$ obtaining

$$\left.\frac{\partial\Theta}{\partial T}\right|_{S_\mathrm{A},p} = \eta_T\left(S_\mathrm{A},t,p\right)\left.\frac{\partial\Theta}{\partial\eta}\right|_{S_\mathrm{A}} = -\left(T_0+\theta\right)\frac{g_{TT}\left(S_\mathrm{A},t,p\right)}{c_p^0} = \frac{\left(T_0+\theta\right)}{\left(T_0+t\right)}\frac{c_p\left(S_\mathrm{A},t,p\right)}{c_p^0}, \qquad \text{(A.15.3)}$$

where the second part of this equation has used Eqn. (A.12.4) for $\left.\Theta_\eta\right|_{S_\mathrm{A}}$. This is then used to obtain the desired expression Eqn. (2.18.3) for $\alpha^\Theta$ as follows

$$\alpha^\Theta = \frac{1}{v}\left.\frac{\partial v}{\partial\Theta}\right|_{S_\mathrm{A},p} = \frac{1}{v}\left.\frac{\partial v}{\partial T}\right|_{S_\mathrm{A},p}\left(\left.\frac{\partial\Theta}{\partial T}\right|_{S_\mathrm{A},p}\right)^{-1} = -\frac{g_{TP}\left(S_\mathrm{A},t,p\right)}{g_P\left(S_\mathrm{A},t,p\right)}\frac{c_p^0}{\left(T_0+\theta\right)g_{TT}\left(S_\mathrm{A},t,p\right)}. \qquad \text{(A.15.4)}$$

In order to derive Eqn. (2.19.2) for $\beta^\theta$ we first need an expression for $\left.\partial\theta/\partial S_\mathrm{A}\right|_{T,p}$. This is found by differentiating with respect to Absolute Salinity the entropy equality $\eta\left(S_\mathrm{A},t,p\right) = \eta\left(S_\mathrm{A},\theta\left[S_\mathrm{A},t,p,p_\mathrm{r}\right],p_\mathrm{r}\right)$ which defines potential temperature, obtaining

$$\begin{aligned}\left.\frac{\partial\theta}{\partial S_\mathrm{A}}\right|_{T,p} &= \left.\theta_\eta\right|_{S_\mathrm{A}}\left[\eta_{S_\mathrm{A}}\left(S_\mathrm{A},t,p\right) - \eta_{S_\mathrm{A}}\left(S_\mathrm{A},\theta,p_\mathrm{r}\right)\right] \\ &= \frac{\left(T_0+\theta\right)}{c_p\left(S_\mathrm{A},\theta,p_\mathrm{r}\right)}\left[\mu_T\left(S_\mathrm{A},\theta,p_\mathrm{r}\right) - \mu_T\left(S_\mathrm{A},t,p\right)\right] \qquad \text{(A.15.5)} \\ &= \left[g_{S_\mathrm{A}T}\left(S_\mathrm{A},t,p\right) - g_{S_\mathrm{A}T}\left(S_\mathrm{A},\theta,p_\mathrm{r}\right)\right]\Big/ g_{TT}\left(S_\mathrm{A},\theta,p_\mathrm{r}\right),\end{aligned}$$

where Eqns. (A.12.5) and (A.12.7) have been used with a general reference pressure $p_\mathrm{r}$ rather than with $p_\mathrm{r} = 0$. By differentiating $\rho = \tilde{\rho}\left(S_\mathrm{A},\theta\left[S_\mathrm{A},t,p,p_\mathrm{r}\right],p\right)$ with respect to Absolute Salinity it can be shown that (Gill (1982), McDougall (1987a))

$$\beta^\theta = \frac{1}{\rho}\left.\frac{\partial\rho}{\partial S_\mathrm{A}}\right|_{\theta,p} = \frac{1}{\rho}\left.\frac{\partial\rho}{\partial S_\mathrm{A}}\right|_{T,p} + \alpha^\theta\left.\frac{\partial\theta}{\partial S_\mathrm{A}}\right|_{T,p}, \qquad \text{(A.15.6)}$$

and using Eqn. (A.15.5) we arrive at the desired expression Eqn. (2.19.2) for $\beta^\theta$

$$\beta^\theta = -\frac{g_{S_\mathrm{A}P}\left(S_\mathrm{A},t,p\right)}{g_P\left(S_\mathrm{A},t,p\right)} + \frac{g_{TP}\left(S_\mathrm{A},t,p\right)\left[g_{S_\mathrm{A}T}\left(S_\mathrm{A},t,p\right) - g_{S_\mathrm{A}T}\left(S_\mathrm{A},\theta,p_\mathrm{r}\right)\right]}{g_P\left(S_\mathrm{A},t,p\right)g_{TT}\left(S_\mathrm{A},t,p\right)}. \qquad \text{(A.15.7)}$$

Note that the terms in the natural logarithm of the square root of Absolute Salinity cancel from the two parts of the square brackets in Eqns. (A.15.5) and (A.15.7).

In order to derive Eqn. (2.19.3) for $\beta^\Theta$ we first need an expression for $\left.\partial\Theta/\partial S_\mathrm{A}\right|_{T,p}$. This is found by differentiating with respect to Absolute Salinity the entropy equality $\eta\left(S_\mathrm{A},t,p\right) = \hat{\eta}\left(S_\mathrm{A},\Theta\left[S_\mathrm{A},t,p\right]\right)$ obtaining (using Eqns. (A.12.4) and (A.12.8))

$$\begin{aligned}\left.\frac{\partial\Theta}{\partial S_\mathrm{A}}\right|_{T,p} &= \left.\Theta_\eta\right|_{S_\mathrm{A}}\left[\eta_{S_\mathrm{A}}\left(S_\mathrm{A},t,p\right) - \left.\hat{\eta}_{S_\mathrm{A}}\right|_\Theta\right] \\ &= \left[\mu\left(S_\mathrm{A},\theta,0\right) - \left(T_0+\theta\right)\mu_T\left(S_\mathrm{A},t,p\right)\right]\Big/ c_p^0 \qquad \text{(A.15.8)} \\ &= \left[g_{S_\mathrm{A}}\left(S_\mathrm{A},\theta,0\right) - \left(T_0+\theta\right)g_{S_\mathrm{A}T}\left(S_\mathrm{A},t,p\right)\right]\Big/ c_p^0.\end{aligned}$$

Differentiating $\rho = \hat{\rho}\left(S_\mathrm{A},\Theta\left[S_\mathrm{A},t,p\right],p\right)$ with respect to Absolute Salinity leads to

$$\beta^\Theta = \frac{1}{\rho}\left.\frac{\partial\rho}{\partial S_\mathrm{A}}\right|_{\Theta,p} = \frac{1}{\rho}\left.\frac{\partial\rho}{\partial S_\mathrm{A}}\right|_{T,p} + \alpha^\Theta\left.\frac{\partial\Theta}{\partial S_\mathrm{A}}\right|_{T,p}, \qquad \text{(A.15.9)}$$

and using Eqn. (A.15.8) we arrive at the desired expression (2.19.3) for $\beta^\Theta$ namely

$$\beta^\Theta = -\frac{g_{S_\mathrm{A}P}\left(S_\mathrm{A},t,p\right)}{g_P\left(S_\mathrm{A},t,p\right)} + \frac{g_{TP}\left(S_\mathrm{A},t,p\right)\left[g_{S_\mathrm{A}T}\left(S_\mathrm{A},t,p\right) - g_{S_\mathrm{A}}\left(S_\mathrm{A},\theta,0\right)/\left(T_0+\theta\right)\right]}{g_P\left(S_\mathrm{A},t,p\right)g_{TT}\left(S_\mathrm{A},t,p\right)}. \qquad \text{(A.15.10)}$$

Note that the terms in the natural logarithm of the square root of Absolute Salinity cancel from the two parts of the square brackets in Eqns. (A.15.8) and (A.15.10).