Notes on the function gsw_CT_second_derivatives(SA,pt)

This function, **gsw_CT_second_derivatives**(SA,pt), evaluates the second derivatives of $\tilde{\Theta}(S_{\mathrm{A}},\theta)$, namely

$$\tilde{\Theta}_{S_{\mathrm{A}}S_{\mathrm{A}}}, \qquad \tilde{\Theta}_{\theta\theta} \qquad \text{and} \qquad \tilde{\Theta}_{S_{\mathrm{A}}\theta}\,. \tag{1}$$

These second derivatives are found by taking finite differences of the first derivatives $\tilde{\Theta}_{S_{\mathrm{A}}}$ and $\tilde{\Theta}_{\theta}$ obtained from **gsw_CT_first_derivatives**(SA,pt) over small increments of Absolute Salinity and potential temperature. The increments of Absolute Salinity are $\pm 0.001\,\mathrm{g\ kg}^{-1}$ and the increments of potential temperature are $\pm 0.01\,{}^{\circ}\mathrm{C}$. If the input Absolute Salinity is less than $0.001\,\mathrm{g\ kg}^{-1}$ care is taken to ensure that the Absolute Salinity is never less than zero in any call to **gsw_CT_first_derivatives**(SA,pt).

The three outputs of this function, **gsw_CT_second_derivatives**(SA,pt), remain well-behaved as the input Absolute Salinity approaches zero, including at $S_{\mathrm{A}} = 0\,\mathrm{g\,kg}^{-1}$.

References

IOC, SCOR and IAPSO, 2010: *The international thermodynamic equation of seawater – 2010: Calculation and use of thermodynamic properties*. Intergovernmental Oceanographic Commission, Manuals and Guides No. 56, UNESCO (English), 196 pp. Available from http://www.TEOS-10.org

Here follows appendix A.12 of the TEOS-10 Manual (IOC *et al.*, 2010).

A.12 Differential relationships between η, θ, Θ and S_{A}

Evaluating the fundamental thermodynamic relation in the forms (A.11.6) and (A.11.12) and using the four boxed equations in appendix A.11, we find the relations

$$\begin{aligned}(T_0+t)\,\mathrm{d}\eta+\mu(p)\,\mathrm{d}S_{\mathrm{A}} &= \frac{(T_0+t)}{(T_0+\theta)}\,c_p(0)\;\mathrm{d}\theta \;+\; \big[\mu(p)-(T_0+t)\,\mu_T(0)\big]\mathrm{d}S_{\mathrm{A}}\\ &= \frac{(T_0+t)}{(T_0+\theta)}\,c_p^0\;\mathrm{d}\Theta \;+\; \left[\mu(p)-\frac{(T_0+t)}{(T_0+\theta)}\,\mu(0)\right]\mathrm{d}S_{\mathrm{A}}\,.\end{aligned} \tag{A.12.1}$$

The quantity $\mu(p)\,\mathrm{d}S_{\mathrm{A}}$ is now subtracted from each of these three expressions and the whole equation is then multiplied by $(T_0+\theta)/(T_0+t)$ obtaining

$$(T_0+\theta)\,\mathrm{d}\eta \;=\; c_p(0)\;\mathrm{d}\theta-(T_0+\theta)\,\mu_T(0)\;\mathrm{d}S_{\mathrm{A}} \;=\; c_p^0\;\mathrm{d}\Theta-\mu(0)\;\mathrm{d}S_{\mathrm{A}}\,. \tag{A.12.2}$$

From this follows all the following partial derivatives between η, θ, Θ and S_{A},

$$\Theta_\theta\big|_{S_{\mathrm{A}}} = c_p(S_{\mathrm{A}},\theta,0)/c_p^0, \qquad \Theta_{S_{\mathrm{A}}}\big|_{\theta} = \big[\mu(S_{\mathrm{A}},\theta,0)-(T_0+\theta)\,\mu_T(S_{\mathrm{A}},\theta,0)\big]/c_p^0, \tag{A.12.3}$$

$$\Theta_\eta\big|_{S_{\mathrm{A}}} = (T_0+\theta)/c_p^0, \qquad \Theta_{S_{\mathrm{A}}}\big|_{\eta} = \mu(S_{\mathrm{A}},\theta,0)/c_p^0, \tag{A.12.4}$$

$$\theta_\eta\big|_{S_{\mathrm{A}}} = (T_0+\theta)/c_p(S_{\mathrm{A}},\theta,0), \qquad \theta_{S_{\mathrm{A}}}\big|_{\eta} = (T_0+\theta)\,\mu_T(S_{\mathrm{A}},\theta,0)/c_p(S_{\mathrm{A}},\theta,0), \tag{A.12.5}$$

$$\theta_\Theta\big|_{S_{\mathrm{A}}} = c_p^0/c_p(S_{\mathrm{A}},\theta,0), \qquad \theta_{S_{\mathrm{A}}}\big|_{\Theta} = -\big[\mu(S_{\mathrm{A}},\theta,0)-(T_0+\theta)\,\mu_T(S_{\mathrm{A}},\theta,0)\big]/c_p(S_{\mathrm{A}},\theta,0), \tag{A.12.6}$$

$$\eta_\theta\big|_{S_{\mathrm{A}}} = c_p(S_{\mathrm{A}},\theta,0)/(T_0+\theta), \qquad \eta_{S_{\mathrm{A}}}\big|_{\theta} = -\mu_T(S_{\mathrm{A}},\theta,0), \tag{A.12.7}$$

$$\eta_{\Theta}\big|_{S_\mathrm{A}} = c_p^0 \big/ \left(T_0 + \theta\right), \qquad\qquad \eta_{S_\mathrm{A}}\Big|_{\Theta} = -\mu\left(S_\mathrm{A},\theta,0\right) \big/ \left(T_0+\theta\right). \qquad \text{(A.12.8)}$$

The three second order derivatives of $\hat{\eta}\left(S_\mathrm{A},\Theta\right)$ are listed in Eqns. (P.14) and (P.15) of appendix P. The corresponding derivatives of $\hat{\theta}\left(S_\mathrm{A},\Theta\right)$, namely $\hat{\theta}_{\Theta}$, $\hat{\theta}_{S_\mathrm{A}}$, $\hat{\theta}_{\Theta\Theta}$, $\hat{\theta}_{S_\mathrm{A}\Theta}$ and $\hat{\theta}_{S_\mathrm{A}S_\mathrm{A}}$ can also be derived using Eqn. (P.13), obtaining

$$\hat{\theta}_{\Theta} = \frac{1}{\tilde{\Theta}_{\theta}}, \quad \hat{\theta}_{S_\mathrm{A}} = -\frac{\tilde{\Theta}_{S_\mathrm{A}}}{\tilde{\Theta}_{\theta}}, \quad \hat{\theta}_{\Theta\Theta} = -\frac{\tilde{\Theta}_{\theta\theta}}{\left(\tilde{\Theta}_{\theta}\right)^3}, \quad \hat{\theta}_{S_\mathrm{A}\Theta} = -\frac{\tilde{\Theta}_{\theta S_\mathrm{A}}}{\left(\tilde{\Theta}_{\theta}\right)^2} + \frac{\tilde{\Theta}_{S_\mathrm{A}}\tilde{\Theta}_{\theta\theta}}{\left(\tilde{\Theta}_{\theta}\right)^3}, \qquad \text{(A.12.9a,b,c,d)}$$

$$\text{and} \quad \hat{\theta}_{S_\mathrm{A}S_\mathrm{A}} = -\frac{\tilde{\Theta}_{S_\mathrm{A}S_\mathrm{A}}}{\tilde{\Theta}_{\theta}} + 2\,\frac{\tilde{\Theta}_{S_\mathrm{A}}}{\tilde{\Theta}_{\theta}}\,\frac{\tilde{\Theta}_{\theta S_\mathrm{A}}}{\tilde{\Theta}_{\theta}} - \left(\frac{\tilde{\Theta}_{S_\mathrm{A}}}{\tilde{\Theta}_{\theta}}\right)^2 \frac{\tilde{\Theta}_{\theta\theta}}{\tilde{\Theta}_{\theta}}, \qquad \text{(A.12.10)}$$

in terms of the partial derivatives $\tilde{\Theta}_{\theta}$, $\tilde{\Theta}_{S_\mathrm{A}}$, $\tilde{\Theta}_{\theta\theta}$, $\tilde{\Theta}_{\theta S_\mathrm{A}}$ and $\tilde{\Theta}_{S_\mathrm{A}S_\mathrm{A}}$ which can be obtained by differentiating the polynomial $\tilde{\Theta}\left(S_\mathrm{A},\theta\right)$ from the TEOS-10 Gibbs function.